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CRITICISMS AND DISCUSSIONS.

HILBERT'S FOUNDATIONS OF GEOMETRY.[1]

Since its appearance in 1899 Hilbert's work on *The Foundations of Geometry* has had a wider circulation than any other modern essay in the realms of pure mathematics. As types of the appreciation which it has received, we may refer to the reviews of Sommer,[2] Poincaré,[3] Halsted,[4] and Hedrick.[5] In each case Hilbert's achievement is enthusiastically praised, as well as (in the latter two cases) the enterprise of the publishers and translator of the English edition. Dr. Hedrick's review in particular is valuable for correcting certain errors of detail in the translation that are of importance to the mathematical reader. Since the purpose of the present article is to point out the relation of Hilbert's work to certain less widely known investigations of the same subject, we can here only add the opinion that the translation is a very competent and useful piece of work.

Study of the foundations of geometry began with no more recent a mathematician than Euclid. In giving form to the hitherto unorganised mass of geometrical knowledge, it was necessary to ask the question: What are the statements that it is necessary to presuppose in order logically to deduce from them the properties of space? Euclid's reply to this conundrum was supposed for many centuries to be the only one possible. The non-Euclidean geometry which arose out of vain attempts to prove the axiom of parallels a consequence of the other axioms was the first break in this tradition. Its development has been brilliant and interesting, but we must summarise it in a list of names: Saccheri (1733), Legendre, Gauss, Schweikart, Lobatschewsky, Bolyai, Riemann, Beltrami, Helmholtz, Cayley, Klein, Lie.

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[1] *The Foundations of Geometry*, D. Hilbert. Authorised translation by E. J. Townsend, Ph. D. Chicago: The Open Court Publishing Company, 1902. 8vo. Pages, vii, 132.

[2] An English translation of Sommer's article appeared in the *Bulletin of the American Mathematical Society*, Vol. VI., p. 287. 1900.

[3] *Bulletin des Sciences Mathématiques*, Vol. XXVI., p. 249, September, 1902.

[4] *The Open Court*, September, 1902.

[5] *Bulletin of the American Mathematical Society*, Vol. IX. (1902), p. 158.

The non-Euclidean geometers, however, confined their attention in the main to the axiom of parallel lines, and it was not till about twenty years ago that an attempt was made to recast the entire system. In 1882, in a book called *Vorlesungen über die neuere Geometrie*, Dr. Moritz Pasch published a system of axioms and showed how it was possible from them to deduce the theorems of geometry. The work of Pasch was considerably simplified and improved by Peano,[1] who attacked the problem with the aid of a ratiocinative calculus. An independent attempt was made by another Italian, Veronese,[2] who studied in particular the continuity of space. Veronese's book, which is somewhat obscure, has received a helpful commentary from T. Levi-Civita.[3]

At the time of Hilbert's investigations, the term "mathematical science" had acquired a precise technical meaning. A mathematical science consists of a body of propositions stated about certain elements in terms of certain relations. The elements in Hilbert's system, for example, are denoted by the words "point," "line," "plane." The relations are indicated by such words as "are situated," "between," "congruent." An element is understood to be quite independent of any mental image that may accompany it in our thinking. It has no properties except such as are stated in the propositions. The axioms are a system of propositions such that:

1. Every proposition of the mathematical science in question can be deduced from the axioms.
2. No axiom can be deduced from the other axioms.

Hilbert arranges his axioms in five groups according to the relations to which they give meaning.

I, 1–7. Axioms of connection (involving the term "are situated").
II, 1–5. Axioms of order (involving the term "between").
III, Axiom of parallels.
IV, 1–6. Axioms of congruence.
V, Axiom of continuity.

There are thus twenty axioms in Hilbert's system.

The sense in which an element is independent of the mental images accompanying it is made clear by the method of proving an axiom to be independent,—i. e., not deducible from the others. This is done simply by exhibiting a self-consistent mathematical science, in which all the axioms except the one in question are valid propositions, whereas that one is not. For if the indicted axiom were deducible from the others, it would be a true proposition of the new science.

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[1] Peano, *I principii de Geometria*, Turin, 1889; "Sui Fondamenti della Geometria," *Rivista di Matematica*, Vol. 4 (1894), pp. 51–90.

[2] Veronese, *Fondamenti di Geometria a più dimensioni e a più speciè de unità rettilinee*. Padua, 1891. German translation by Schepp, Leipzig, 1894.

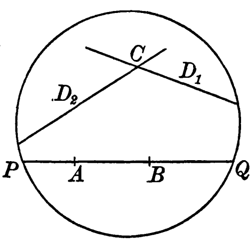
[3] *Memorie della Reale Accademia dei Lincei Roma*, Vol. 7, pp. 91–96, 113–121; 1898.

Take or example the classical case of the Euclidean axiom, Hilbert's III. A mathematical science of the kind required is furnished by the propositions about the points interior to an ordinary sphere. Let the significance in our new science of the words

"point" be interior point of the sphere S (not including the boundary of sphere),

"line" be portion of line interior to the sphere S,

"plane" be portion of plane interior to the sphere S.



The relations of order and connection of "points," "lines," and "planes" are as usual, but on a "line" l (which is the portion of the ordinary line bounded by the points P and Q) the "distance" apart of two points A and B shall be

$$\log\left(\frac{AP.\,BQ}{AQ.\,BP}\right)$$

where AP, etc., are distances in the usual sense.

With a similar convention about the meaning of the word "angle," a very little mathematics suffices to show that all the axioms of the groups I, II, IV, V (and of course the theorems deducible from them) are valid propositions about our new "points," "lines," etc. For example, "space" is infinite, for if A is moved toward Q, the "distance AB," as defined, increases indefinitely.

Moreover, unless the Euclidean geometry is at war with itself, these propositions contain no self-contradiction; for if they did, on translating them back into the corresponding properties of the sphere the contradiction would remain and therefore the Euclidean geometry would contain self-contradictions. Now if the parallel axiom were deducible from I, II, IV, V, we should be able to obtain the proposition that through a point C there is one and *only* one "line" which does not meet the "line" AB. It is plain at once that there are *many* "lines" (CD_1, CD_2, for example) that do not meet AB. The proof is therefore complete that III cannot be deduced from I, II, IV, V.

It was generally supposed[1] till the researches of Peano and Hilbert became known, that even though this kind of analysis could be applied to the parallel axiom, there might be other axioms to which it would not apply. It has by this time, however, been extended to all the axioms both of arithmetic and geometry. The mental images that are applied to the words "point" and "line," etc., in the bizarre mathematical sciences that have been used for this purpose vary from the letters of the alphabet to more or less complicated mathematical curves. The possibility is also recognised of using different mental images in the same science. For example, a distortion can be conceived to be applied to space in such a way that

[1] Cf. Poincaré, *Nature*, Feb. 25, 1892; Russell, *The Foundations of Geometry*, Introduction, p. 6, 1897, Cambridge University Press.

all the lines are bent without altering their mutual relations. It is evident that if we think the geometrical relations in terms of the bent lines our images are changed without affecting the form of the axioms or the method of deduction. Theoretically, at least, the deductions could be made without any reference to their content by the use of a ratiocinative calculus like that of Peano or a Jevons logical machine.

The essential point is the recognition of the processes of logic, as separate from the mental imagery of the "elements." For this reason it is perhaps well to use the word "symbol," instead of "element" and "relation" as that, in the formal process, which answers to a basal concept of a science. The symbols of Hilbert's science, accordingly, are "points," "lines," "between," etc. The symbolic point of view also makes it possible to state in simple language what is meant by a definition.

Theoretically, all the propositions of a mathematical science can be stated in terms of the fundamental "elements" and their relations. As a rough example—we should never need to speak of "circles" but only of "sets of points in a plane equally distant from a given point." It is however convenient to introduce the term "circle" by a definition. Again, in arithmetic we may have the *definition*, $3+1=4$. In general, we mean by a definition an agreement that permits us to substitute a simple term or symbol for more complex terms or symbols. The only symbols that are undefined are the basal elements.

The question immediately arises, is it perhaps possible to define some of the elements in terms of the other elements? In that case our system of assumptions is clearly not as simple as possible, even if no one of the axioms is deducible from the others. We have therefore, in addition to the problem of determining a set of independent axioms, the analogous problem of determining an irreducible set of elements, i. e., a system of elements such that no one of them can be defined in terms of the others.[1]

The two problems are evidently interrelated, and Hilbert's geometry furnished a good example of that fact. Axiom I, 3 reads as follows:

I, 3: Three points A, B, C, not situated in the same straight line, always completely determine a plane α. We write $ABC=\alpha$.

This axiom was stated by Prof. F. Schur[2] of Karlsruhe to be deducible from the rest, but Professor E. H. Moore of Chicago, in the paper cited below, proved that such a statement is incorrect under the literal interpretation of Hilbert whereby a

[1] This problem has been attacked and stated in a general, though somewhat different, form by Professor A. Padoa of the University of Rome. See *Bibliothèque du Congrès International de Philosophie, III. Logique et Histoire des Sciences*. Page 309, 1901. The method he proposes seems hardly adequate.

[2] Schur, "Ueber die Grundlagen der Geometrie," *Mathematische Annalen*, Vol. LV. (1901), pp. 265–292.

plane is made one of the undefined elements. The symbol "plane" is capable of definition in terms of "point" and "line," but unless this definition is admitted the axiom I, 3 is not a consequence of the other axioms. We thus have at hand a case showing how a system of axioms may be non-redundant while the system of elements is not irreducible.

It is along this direction that the severest criticism of Hilbert's work can be made. While his axioms are in the main independent, his system of elements is far from irreducible. In the axioms of the groups I and II there are three elements, points, lines, and planes, and two relations, represented by the phrases "are situated" and "between." It may be here stated, leaving the proof for a more technical article, that all these elements and relations can be defined in terms of "points" and "betweenness."[1]

Hilbert himself defines the relations of groups III and V in terms of the other relations. But he does not make use of the fact pointed out some twenty years ago by Klein,[2] that the congruence relation of group IV (which amounts to the same thing as rigid motion) can also be defined.

With respect to the congruence axioms, Hilbert makes[3] a curiously misleading statement:

> "We shall show the independence of the axioms of congruence by demonstrating that axiom IV, 6 . . . . cannot be deduced from the remaining axioms I, II, III, IV, 1–5, V, by any logical process of reasoning."

If we agree that congruence is an *element* it is obvious without further proof that the axioms of IV cannot be deduced from I, II, III, V, because congruence is not mentioned in I, II, III, V. What Hilbert really does, however, is to exhibit a mathematical science in which I, II, III, V are satisfied, and in which there is given a definition of congruence that verifies IV, 1–5, without verifying IV, 6. The effect of this is merely to demonstrate that a certain relation has the first five without the sixth property of congruence. The possibility still remains that by means of a suitable definition all the propositions of IV can be logically deduced.

The independence of his axioms was not fully investigated by Hilbert himself, but further research has revealed only one axiom which is capable of being deduced from the other axioms if the elements are retained precisely as Hilbert gave

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[1] For other ways of reducing to two the number of "elements" see Padoa "Un nouveau systèm de définitions pour la géométrie euclidienne." *2d. congrès international des mathématiciens*, 1900, p. 353. Reference is there made by Padoa to related work of Pieri.

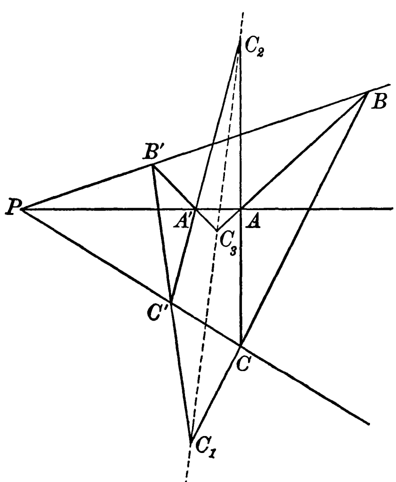
[2] Klein, *Math. Ann.*, Vols. IV., p. 573, and VI., p. 271 (1871–1873). It is important to add that the definition of congruence is not unique in the same sense as is the definition of "plane." In other words, by the Klein definition, rigid motion is a special case of a more general kind of "transformation of space."

[3] Art. 11, p. 32 of the translation.

them. This is axiom II, 4, the discovery of whose redundance was made a little over a year ago by Professor Moore.[1]

It is important to remark after the criticisms just cited that the imperfections of his axioms need not impair the importance of Hilbert's contribution to geometry. His work on axioms, indeed, can hardly be called more original or elegant than that of the Peano school in Italy. He has, however, by an attractive form of presentation drawn the attention of the entire mathematical world to one of the most fertile fields of investigation both for the technical mathematician and for the philosopher.

Besides this, Hilbert has extended the scope of his investigations so as to apply to any theorem whatever and not merely to the axioms. Take for example the famous theorem of Desargues:



> If two triangles are so situated in a plane that the lines joining their corresponding vertices AA', BB', CC', meet in a point P and the corresponding sides AB, $A'B'$, etc., meet in three points, C_1, C_2, C_3; then C_1, C_2, C_3 are collinear.

This theorem evidently has not to do with the metrical properties of the triangle. It becomes obviously true if the two triangles are not in the same plane, for then the corresponding sides can meet only in the line of intersection of the two planes. Nevertheless the theorem, when the two triangles are in the same plane, can *not* be demonstrated in a non-metrical way unless a construction is used which involves points outside the plane of the two triangles. This fact had long been suspected but never, before Hilbert's time, proved. The method of proof is an extension of that used to prove the independence of an axiom. A mathematical science is exhibited in which all the non-metrical axioms about the plane are valid, but in which it is not assumed that there are points outside the plane. In this mathematical science[2] the Desargues

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[1] E. H. Moore, "On the Projective Axioms of Geometry," *Transactions of the American Mathematical Society*, Vol. III., p. 142, January, 1902. A somewhat simpler proof, due to Mr. R. L. Moore, was published in an article on "The Betweenness Assumptions," *American Mathematical Monthly*, April, 1902.

[2] A simpler example of a non-Desarguean science than that of Hilbert is given by Dr. F. R. Moulton, *Transactions of the American Mathematical Society*, Vol. III. (1902), p. 192.

proposition is not a valid theorem, and therefore cannot be deduced from the axioms mentioned.

The extension of this method to the Desargues theorem and to many others is. I believe, the main achievement of Hilbert and of his students. In a sense it is the final extension to geometry of a tendency that has been powerful in the realms of analysis for the last half century. This is the desire, on the one hand, for perfectly rigorous logic in demonstration, and on the other hand to state theorems in the form of necessary and sufficient conditions.

A necessary and sufficient statement is of this form : "If A, then B, and if B, then A." Both the direct statement and the converse of a theorem are to hold. Stated in this language, the problem of the foundations of geometry is to find a system of axioms that is necessary and sufficient for geometry,—necessary, in that no axiom can be dispensed with, and sufficient for the deduction of the whole system of geometrical knowledge. In like manner the basal elements are to be necessary and sufficient for the definition of every other symbol of the science. The investigation of each theorem ought to be such that we can say that 'axioms a, b, c are necessary and sufficient for the validity of theorem D.' Finally, of each definition we should like to say that it is sufficient to determine the thing defined, whereas each symbol entering into it is necessary for such determination. The formulation of a science with this degree of necessity and sufficiency is the ideal of the investigators of the foundations of geometry.

OSWALD VEBLEN.

THE UNIVERSITY OF CHICAGO, December 1, 1902.